Summary of Dual and Quaternion Mathematics for Kinematics

Dual and quaternion mathematics are useful tools for describing and computing kinematics. Their theoretical foundations are well documented in the literature of theoretical kinematics, and their application to computer animation and computational kinematics is becoming more popular. While their application might be very straightforward, their descriptions in the literature are on the one hand completely fogged in by theoretical surroundings or on the other hand by seemingly deliberate obfuscation. Not the least of the problem is that is that workers with various foundations and motives use various parameterizations and notations to express the same things.

So it is the intent of this paper to present in one place and in what I hope to be consistent parameterization and notation, a summary of the math of dual numbers, quaternions, and dual quaternions, and the operations that make them useful for doing kinematics. This is not a tutorial either in mathematics or in kinematics. It's just a collection of notes I assembled on my path of tears trying to decipher this business.

In each of the sections that follow, the equations and functions shown in the shaded blocks are Mathcad 2000 implementations of the formulations.

If I've succeeded to the extent that you can review it, I'd welcome corrections or suggestions at klingener@BrockEng.com.

CONTENTS

1.0 Dual numbers

1.1 Parameterization

$\hat{d} = a + e \cdot b, \quad e^2 = e^3 = \ldots = 0$ a, b are real number parameters

The dual number $\hat{d}$ (sometimes called a 'duplex' number) is composed of a 'real' or 'primary' part, a, and a 'dual' part b.

$$\text{dual}(a,b) := \left| \begin{array}{l} ret_0 \leftarrow a \\ ret_1 \leftarrow b \\ ret \end{array} \right.$$

$$d1 := \text{dual}\left(\frac{\pi}{3}, 2\right) \qquad d2 := \text{dual}\left(\frac{\pi}{7}, 5\right)$$

1.2 Addition

$$\hat{d}_1 + \hat{d}_2 = (a_1 + a_2) + e \cdot (b_1 + b_2) = \hat{d}_2 + \hat{d}_1$$

$$\text{dual\_add}(d1, d2) := \text{dual}(d1_0 + d2_0, d1_1 + d2_1)$$

$$\text{dual\_add}(d1, d2) = \begin{pmatrix} 1.496 \\ 7 \end{pmatrix}$$

1.3 Multiplication

$$\hat{d}_1 \cdot \hat{d}_2 = (a_1 \cdot a_2) + e \cdot (b_1 \cdot a_2 + a_1 \cdot b_2) = \hat{d}_2 \cdot \hat{d}_1$$

$$\text{dual\_mul}(d1, d2) := \text{dual}(d1_0 \cdot d2_0, d1_1 \cdot d2_0 + d1_0 \cdot d2_1)$$

$$\text{dual\_mul}(d1, d2) = \begin{pmatrix} 0.47 \\ 6.134 \end{pmatrix}$$

1.4 Division

$$\hat{d}_1 / \hat{d}_2 = (a_1 / a_2) + e \cdot (a_2 \cdot b_1 - a_1 \cdot b_2) / a_2^2 \qquad a_2 \neq 0$$

$$\text{dual\_div}(d1, d2) := \left| \begin{array}{l} a \leftarrow \dfrac{d1_0}{d2_0} \\ b \leftarrow \dfrac{d2_0 \cdot d1_1 - d1_0 \cdot d2_1}{(d2_0)^2} \\ \text{dual}(a, b) \end{array} \right.$$

$$\text{dual\_div}(d1, d2) = \begin{pmatrix} 2.333 \\ -21.539 \end{pmatrix}$$

1.5 Reciprocal

$$1/\hat{d} \quad = (1/a) - e\ (b/a^2) = (1/a)(1 - e\ (b/a)) \qquad a \neq 0$$
$$= (a - e\ b)/a^2$$

$\text{dual\_recip}(d) := \text{dual}\left[\frac{1}{d_0}, \frac{-d_1}{(d_0)^2}\right]$ $\text{dual\_recip}(d1) = \begin{pmatrix} 0.955 \\ -1.824 \end{pmatrix}$

1.6 Power

$$\hat{d}^n = (a + e \cdot b)^n = a^n + e \cdot n \cdot b \cdot a^{n-1}$$

$$\text{dual\_power}(d, n) := \begin{vmatrix} a \leftarrow (d_0)^n \\ b \leftarrow n \cdot d_1 \cdot (d_0)^{(n-1)} \\ \text{dual}(a, b) \end{vmatrix}$$

$\text{dual\_power}(d1, 3) = \begin{pmatrix} 1.148 \\ 6.58 \end{pmatrix}$

$$\sqrt{\hat{d}} = \sqrt{a} + e \cdot \frac{b}{2\sqrt{a}} = \sqrt{a}\left(1 + e \cdot \frac{b}{2 \cdot a}\right) \qquad a > 0$$

$\text{dual\_sqrt}(d) := \text{dual}\left(\sqrt{d_0}, \frac{d_1}{2 \cdot \sqrt{d_0}}\right)$ $\text{dual\_sqrt}(d1) =$ ▮

1.7 Functions

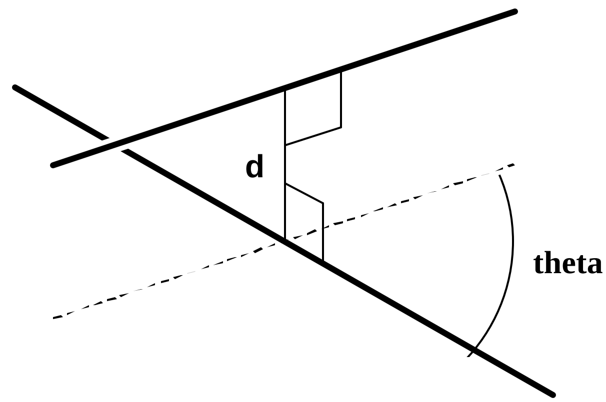
$$f(d) = f(a) + e \cdot b \cdot f'(a)$$
$$\sin(d) = \sin(a) + e \cdot b \cdot \cos(a)$$
$$\cos(d) = \cos(a) - e \cdot b \cdot \sin(a)$$
$$\cot(d) = \cot(a) - e \cdot b / \sin^2(a) \qquad \sin(a) \neq 0$$

$\text{dual\_sin}(d) := \text{dual}(\sin(d_0), d_1 \cdot \cos(d_0))$ $\text{dual\_sin}(d1) = \begin{pmatrix} 0.866 \\ 1 \end{pmatrix}$ ▮

$\text{dual\_cos}(d) := \text{dual}(\cos(d_0), -d_1 \cdot \sin(d_0))$ $\text{dual\_cos}(d1) = \begin{pmatrix} 0.5 \\ -1.732 \end{pmatrix}$

1.8 Application to kinematics - screw displacement definition

Dual numbers are particularly useful for expression of dual angles, which are, in turn, useful for expressing the spatial relationship between skew lines in space. Skew (non-intersecting) straight lines in space are separated by a perpendicular distance, *d*, and the projection of one line onto the other along that perpendicular forms an angle, q. The dual angle describing the relationship is $\hat{q} = q + e \cdot d$.



The dual angle between *intersecting lines* has a 'real part' equal to the angle between them and a 'dual part' equal to zero.

The dual angle between *parallel lines* has a 'real part' equal to zero and a 'dual part' equal to the perpendicular distance between them..

2.0 Dual Vectors

2.1 Parameterization

$$\hat{\boldsymbol{V}} = \vec{\boldsymbol{v}} + e \cdot \vec{\boldsymbol{w}}, \quad \vec{\boldsymbol{v}} \text{ and } \vec{\boldsymbol{w}} \text{ are Euclidean vectors in 3-space}$$

$$\vec{\boldsymbol{v}} = (v_x, v_y, v_z) \quad \vec{\boldsymbol{w}} = (w_x, w_y, w_z)$$

2.2 Addition

$$\hat{\boldsymbol{V}}_1 + \hat{\boldsymbol{V}}_2 = \vec{\boldsymbol{v}}_1 + \vec{\boldsymbol{v}}_2 + e \cdot (\vec{\boldsymbol{w}}_1 + \vec{\boldsymbol{w}}_2)$$

2.3 Multiplication by a dual number

$$\hat{d} \cdot \hat{\boldsymbol{V}} = (a + e \cdot b) \cdot (\vec{\boldsymbol{v}} + e \cdot \vec{\boldsymbol{w}}) = a \cdot \vec{\boldsymbol{v}} + e \cdot (a \cdot \vec{\boldsymbol{w}} + b \cdot \vec{\boldsymbol{v}})$$

2.4 Dual scalar (dot) product

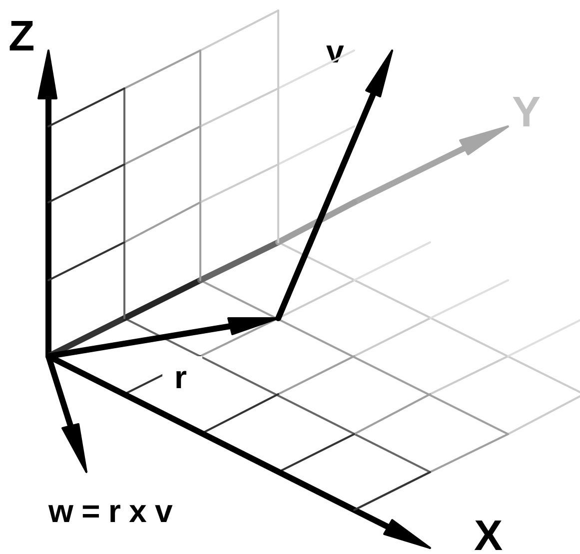
$$\hat{V}_1 \bullet \hat{V}_2 = (\vec{v}_1 + e \cdot \vec{w}_1) \bullet (\vec{v}_2 + e \cdot \vec{w}_2)$$
$$= (\vec{v}_1 \bullet \vec{v}_2 + e \cdot (\vec{v}_1 \bullet \vec{w}_2 + \vec{w}_1 \bullet \vec{v}_2) = \hat{V}_2 \bullet \hat{V}_1$$

2.5 Dual vector (cross) product

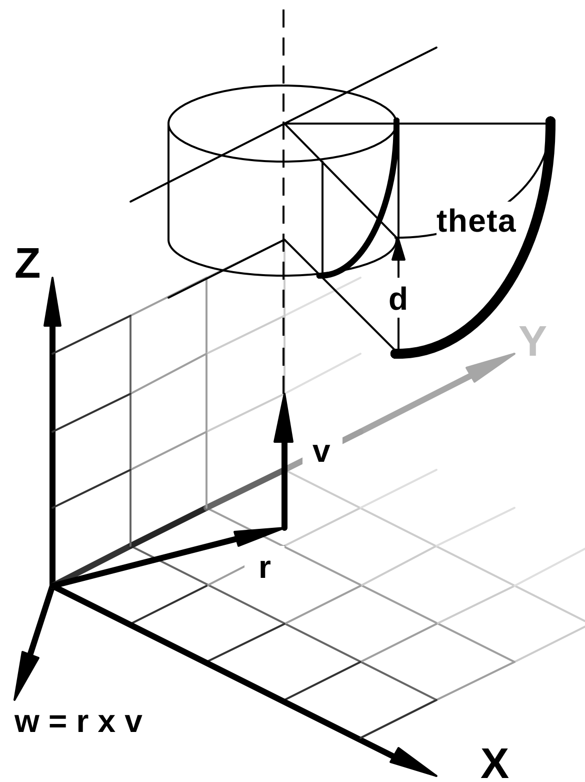
$$\hat{V}_1 \times \hat{V}_2 = (\vec{v}_1 + e \ \vec{w}_1) \times (\vec{v}_2 + e \cdot \vec{w}_2)$$
$$= (\vec{v}_1 \times \vec{v}_2 + e \cdot (\vec{v}_1 \times \vec{w}_2 + \vec{w}_1 \times \vec{v}_2) \neq \hat{V}_2 \times \hat{V}_1$$

2.6 Application to kinematics - screw axis definition

Dual vectors (or bivectors) are useful for expressing a screw displacement axis. The axis vector (the direction of the axis of the screw rotation) is the 'real' part of the dual vector, and the moment of the axis vector about the origin (a measure of the spatial distance from the origin to the line of action of the axis vector) is the 'dual' part of the dual vector.



The bivector B = v + ε w is a vector with magnitude and direction defined by vector v, constrained to act anywhere along a line in space with the same direction as v and passing through the point defined by the position vector r. The primary or real part of B is v, and the dual part is w = r x v. The components of B = (v_x, v_y, v_z, w_x, w_y, w_z) are the Plücker coordinates of the line.



Any displacement of a rigid body in space can be reduced to a screw displacement. A rigid body undergoes a screw displacement when it turns a dual angle about a bivector (dual vector) axis. The *orientation* in space of the axis is the real part of the bivector; the *position* of the axis is defined by the dual part of the bivector; the *angle* through which the body rotates is the real part of the dual angle; the *translation* of the body in the direction of the axis is the dual part of the angle.

3.0 Quaternions

3.1 Parameterization

McCarthy $Q = c_0 + c_1 \cdot i + c_2 \cdot j + c_3 \cdot k$

$$i \cdot i = j \cdot j = k \cdot k = -1,$$

$$i \cdot j = -j \cdot i = k, \quad j \cdot k = -k \cdot j = i, k \cdot i = -i \cdot k = j$$

Bottema & Roth $Q = (c_0, c_1, c_2, c_3)$

Lamb $Q = (w, x, y, z)$ real numbers

Vector notation $Q = (s, \vec{v})$ s a real number, $\vec{v}$ a 3 - vector

Java 3D API Quat4 * $Q = (x, y, z, w)$ same as Lamb but in different order

3.0 Quaternions

This summary $Q = \{w, \{x, y, z\}\}$ and $(s, \vec{v})$

```
quat(w,x,y,z) := | ret0 ← w
                 | ret1 ← x
                 | ret2 ← y
                 | ret3 ← z
                 | ret
```

```
quat_vec(x,y,z) := quat(0,x,y,z)
```

3.2 Representations

real number, r: $\{r, \{0, 0, 0\}$

complex number $z = x + iy$: $\{x, \{y, 0, 0\}\}$

vector quaternion $\vec{V}$: $\{0, \{v_x, v_y, v_z\}\}$

spherical(rotation) quaternion: $\{\cos(q/2), \{\sin(q/2)(s_x, s_y . s_z)\}\}$

where q = rotation angle and $\vec{s}$ is the rotation axis unit vector

3.3 Addition

$$
\begin{aligned}
Q_1 + Q_2 &= \{w_1 + w_2, \{x_1 + x_2, y_1 + y_2, z_1 + z_2\}\} \\
&= Q_2 + Q_1 \\
&= (s_1 + s_2, \vec{\boldsymbol{v}}_1 + \vec{\boldsymbol{v}}_2)
\end{aligned}
$$

```
quat_add(q1,q2) := | for  i ∈ 0.. 3
                   |    ret_i ← q1_i + q2_i
                   | ret
```

3.4 Multiplication

$$
\begin{aligned}
Q_1 \cdot Q_2 = \quad & \{w_1 \cdot w_2 - x_1 \cdot x_2 - y_1 \cdot y_2 - z_1 \cdot z_2, \{ \\
& w_1 \cdot x_2 + x_1 \cdot w_2 + y_1 \cdot z_2 - z_1 \cdot y_2 , \\
& w_1 \cdot y_2 + y_1 \cdot w_2 + z_1 \cdot x_2 - x_1 \cdot z_2 , \\
& w_1 \cdot z_2 + z_1 \cdot w_2 + x_1 \cdot y_2 - y_1 \cdot x_2 \}\} \\
= \quad & (s_1 \cdot s_2 - \vec{\boldsymbol{v}}_1 \bullet \vec{\boldsymbol{v}}_2 , s_1 \cdot \vec{\boldsymbol{v}}_2 + s_2 \cdot \vec{\boldsymbol{v}}_1 + \vec{\boldsymbol{v}}_1 \times \vec{\boldsymbol{v}}_2)
\end{aligned}
$$

The quaternion product between two vector quaternions has the following form:

$$V_1 \cdot V_2 = \{(0, \vec{\boldsymbol{v}}_1) \cdot (0, \vec{\boldsymbol{v}}_2)\} = \{-\vec{\boldsymbol{v}}_1 \bullet \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_1 \times \vec{\boldsymbol{v}}_2\}$$

$$V_2 \cdot V_1 = (V_1 \cdot V_2)^*$$

```
quat_mul(q1,q2) := | ret_0 ← q1_0 · q2_0 − q1_1 · q2_1 − q1_2 · q2_2 − q1_3 · q2_3
                   | ret_1 ← q1_0 · q2_1 + q1_1 · q2_0 + q1_2 · q2_3 − q1_3 · q2_2
                   | ret_2 ← q1_0 · q2_2 + q1_2 · q2_0 + q1_3 · q2_1 − q1_1 · q2_3
                   | ret_3 ← q1_0 · q2_3 + q1_3 · q2_0 + q1_1 · q2_2 − q1_2 · q2_1
                   | ret
```

```
quat_mul_scalar(q,s) := | for i ∈ 0..3
                        |   ret_i ← q_i · s
                        | ret
```

3.5 Conjugate

$$
\begin{aligned}
Q^* \quad = \quad & \{w, \{-x, -y, -z\}\} \\
= \quad & (s, -\vec{\boldsymbol{v}})
\end{aligned}
$$

The conjugate of a vector quaternion $V = (0, \vec{\boldsymbol{v}})$ is $V^* = (0, -\vec{\boldsymbol{v}})$

$$V^* = -V$$

```
quat_conj(q) := quat(q_0, −q_1, −q_2, −q_3)
```

3.6 Norm

$$N(Q) = Q \cdot Q^* = (w^2 + x^2 + y^2 + z^2)$$
$$= (s^2 + \vec{\boldsymbol{v}} \bullet \vec{\boldsymbol{v}})$$

The norm of a vector quaternion $V = (0, \vec{\boldsymbol{v}})$ is $N(V) = V \bullet V^* = \vec{\boldsymbol{v}} \bullet \vec{\boldsymbol{v}}$

$\text{quat\_norm}(q) := \text{quat\_mul}(q, \text{quat\_conj}(q))_0$

$\text{quat\_normalize}(q) := \frac{q}{|q|}$

$\text{quat\_unit\_vec}(x, y, z) := \text{quat\_normalize}(\text{quat\_vec}(x, y, z))$

3.7 Inverse

If $A = (a, \vec{\boldsymbol{a}})$, $Q = (s, \vec{\boldsymbol{v}})$, find A such that $A \cdot Q = (1, \vec{\boldsymbol{0}})$

$$a \cdot \vec{\boldsymbol{v}} + s \cdot \vec{\boldsymbol{a}} + \vec{\boldsymbol{a}} \times \vec{\boldsymbol{v}} = 0$$
$$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{v}} = 0 \Rightarrow \quad \vec{\boldsymbol{a}} = k \cdot \vec{\boldsymbol{v}}$$
$$a \cdot \vec{\boldsymbol{v}} + s \cdot k \cdot \vec{\boldsymbol{v}} = 0 \Rightarrow a = -k \cdot s$$
$$a \cdot s - \vec{\boldsymbol{a}} \bullet \vec{\boldsymbol{v}} = 1$$
$$-k \cdot s^2 - k \cdot \vec{\boldsymbol{v}} \bullet \vec{\boldsymbol{v}} = 1 \Rightarrow -k \cdot N(Q) = 1, k = -1 / N(Q)$$
$$a = s / N(Q), \vec{\boldsymbol{a}} = -\vec{\boldsymbol{v}} / N(Q)$$
$$A = Q^{-1} = Q^* / N(Q)$$

If V is a vector quaternion $= (0, \vec{\boldsymbol{v}})$, then $V^{-1} = (0, -\vec{\boldsymbol{v}}) / \vec{\boldsymbol{v}} \bullet \vec{\boldsymbol{v}}$

If Q is a unit quaternion ($N(Q) = 1$) then $Q^{-1} = Q^*$

$\text{quat\_inv}(q) := \frac{\text{quat\_conj}(q)}{\text{quat\_norm}(q)}$

3.8 Applications in kinematics - spherical transformations

A spherical (rotation) transformation can be expressed by the form:

$X = R \cdot x \cdot R^*$

where X and x are vector quaternions (with a form shown in section 3.2) representing points in fixed and moving reference frames respectively, and R is a rotation quaternion (with a form shown in section 3.2), and R^* is its conjugate.

This transformation is equivalent to the common matrix transformation

$$\begin{vmatrix} X \\ 1 \end{vmatrix} = \begin{vmatrix} R & 0 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} x \\ 1 \end{vmatrix}$$

where, here, X and x are 3- vectors, and R is the 3×3 rotation matrix.

$$\text{quat\_rot}(t, n) := \left| \begin{array}{l} s \leftarrow \sin\left(\frac{t}{2}\right) \\ n \leftarrow \frac{n}{|n|} \\ \text{quat}\left(\cos\left(\frac{t}{2}\right), s \cdot n_1, s \cdot n_2, s \cdot n_3\right) \end{array} \right.$$

quat_transform(Q, x) := quat_mul(quat_mul(Q, x), quat_conj(Q))

4.0 Dual Quaternions

4.1 Parameterization

$$\hat{Q} = Q + e \cdot Q'$$

$Q =$ "real" quaternion part, $Q' =$ "dual" quaternion part

$$= \{w\{x, y, z\}\} + e \cdot \{w'\{x', y', z'\}\}$$

Both Bottema & Roth and McCarthy use parameterizations that, in this context, look like $Q' = Q''Q$. As a consequence,

$$\hat{Q} = (1 + e\ Q'') \cdot Q$$

In kinematics, Q, Q', and Q'' have special forms (discussed later) that simplify manipulations.

```
dualquat(w, x, y, z, w0, x0, y0, z0) := | ret[0,0] ← w
                                        | ret[1,0] ← x
                                        | ret[2,0] ← y
                                        | ret[3,0] ← z
                                        | ret[0,1] ← w0
                                        | ret[1,1] ← x0
                                        | ret[2,1] ← y0
                                        | ret[3,1] ← z0
                                        | ret
```

4.2 Representations

real number r:	$\{r, \{0, 0, 0\}\} + e \cdot \{0, \{0, 0, 0\}\}$
complex number $z = x + iy$:	$\{x, \{y, 0, 0\}\} + e \cdot \{0, \{0, 0, 0\}\}$
dual number $d = a + e \cdot b$:	$\{a, \{0, 0, 0\}\} + e \cdot \{b, \{0, 0, 0\}\}$
quaternion Q:	$\{w, \{x, y, z\}\} + e \cdot \{0, \{0, 0, 0\}\}$
dual vector quaternion:	$\{0, \{v_x, v_y, v_z\}\} + e \cdot \{0, \{0, 0, 0\}\}$
dual bivector quaternion	$\{0, \{v_x, v_y, v_z)) + e \cdot \{0, \{w_x, w_y, w_z\}\}$

```
dualquat_quat(qr, qd) := | for i ∈ 0..3
                         |   ret[i,0] ← qr[i]
                         | for i ∈ 0..3
                         |   ret[i,1] ← qd[i]
                         | ret
```

```
dualquat_vec(x, y, z) := dualquat(0, x, y, z, 0, 0, 0, 0)
```

```
dualquat_bivect(v1, v2, v3, w1, w2, w3) := dualquat(0, v1, v2, v3, 0, w1, w2, w3)
```

```
dualquat_vec_rad(x, y, z, rx, ry, rz) := | V ← quat_vec(x, y, z)
                                         | R ← quat_vec(rx, ry, rz)
                                         | dualquat_quat(V, quat_cross(R, V))
```

```
dualquat_getReal(q) := | for i ∈ 0..3
                       |   ret[i] ← q[i,0]
                       | ret
```

```
dualquat_getDual(q) := | for i ∈ 0..3
                       |   ret[i] ← q[i,1]
                       | ret
```

```
dualquat_getRad(q) := | V ← dualquat_getReal(q)
                      | W ← dualquat_getDual(q)
                      | quat_mul(V, W) / quat_norm(V)
```

4.2 Addition

$$\begin{aligned}\hat{Q}_1 + \hat{Q}_2 &= (Q_1 + e \cdot Q_1') + (Q_2 + e \cdot Q'_2) \\ &= (Q_1 + Q_2 + e \cdot (Q_1' + Q_2')\end{aligned}$$

```
dualquat_add(q1, q2) := | for j ∈ 0..1
                        |   for i ∈ 0..3
                        |     ret[i,j] ← q1[i,j] + q2[i,j]
                        | ret
```

4.3 Multiplication

$$\hat{Q}_1 \cdot \hat{Q}_2 = (Q_1 + e \cdot Q_1') \cdot (Q_2 + e \cdot Q_2')$$
$$= Q_1 \cdot Q_2 + e \cdot (Q_1 \cdot Q_2' + Q_1' \cdot Q_2)$$

The product of two dual vector quaternions has the following form:

$$\hat{V}_1 \cdot \hat{V}_2 = (\{0, \{\vec{\mathbf{v}}_1\}\} + e \cdot \{0, \{0\}\}) \cdot (\{0, \{\vec{\mathbf{v}}_2\}\} + e \cdot \{0, \{0\}\})$$
$$= (\{-\vec{\mathbf{v}}_1 \bullet \vec{\mathbf{v}}_2, \{\vec{\mathbf{v}}_1 \times \vec{\mathbf{v}}_2\}\} + e \cdot \{0, \{0\}\})$$

```
dualquat_mul(q1,q2) := | q1R ← dualquat_getReal(q1)
                       | q1D ← dualquat_getDual(q1)
                       | q2R ← dualquat_getReal(q2)
                       | q2D ← dualquat_getDual(q2)
                       | P1 ← quat_mul(q1R,q2R)
                       | P2 ← quat_mul(q1R,q2D)
                       | P3 ← quat_mul(q1D,q2R)
                       | dualquat_quat(P1,quat_add(P2,P3))
```

4.4 Conjugate

$$\hat{Q}^* = Q^* + e \cdot Q'^*$$

If $Q' = Q'' \cdot Q$, then

$$\hat{Q}^* = Q^* + e\ (Q'' \cdot Q)^* = Q^* + e\ Q^* \cdot Q''^* = Q^* \cdot (1 + e\ Q''^*)$$

The conjugate of a dual vector quaternion $\hat{V}^* = -\hat{V}$

```
dualquat_conj(q) := | qR ← dualquat_getReal(q)
                    | qD ← dualquat_getDual(q)
                    | dualquat_quat(quat_conj(qR),quat_conj(qD))
```

4.5 Norm

$$N(\hat{Q}) = \hat{Q}\cdot\hat{Q}$$
$$= (Q + e\ Q')\cdot(Q^* + e\ Q'^*) = Q\cdot Q^* + e\ (Q'\cdot Q^* + Q\cdot Q'^*)$$

If $Q' = Q''\cdot Q$, and Q'' is a vector quaternion $(Q'' = -Q''^*)$, then

$$N(\hat{Q}) = (1 + e\ Q'')\cdot Q\cdot Q^*\cdot(1 + e\ Q''^*)$$
$$= N(Q)\cdot(1 + e\ (Q'' + Q''^*))$$
$$= N(Q)$$

The norm of a dual vector quaternion $\hat{V} = \{\{0,\ \{\vec{\mathbf{v}}\}\} + e\ \cdot 0\}$ is $N(\hat{V}) = \vec{\mathbf{v}} \bullet \vec{\mathbf{v}}$

```
dualquat_norm(q) := dualquat_mul(q, dualquat_conj(q))
```

4.6 Inverse

$$\hat{Q}^{-1} \cdot \hat{Q} = 1 + e \quad 0 = 1$$

set $\hat{Q}^{-1} = A + e \ B,$ and solve for $A, B.$

$$(A + e \cdot B) \cdot (Q + e \cdot Q') = 1 = A \cdot Q + e \cdot (B \cdot Q + A \cdot Q')$$

$$A \cdot Q = 1$$

$$A = Q^{-1}$$

$$(B \cdot Q + A \cdot Q') = 0$$

$$B \cdot Q = -A \cdot Q' = Q^{-1} \cdot Q'$$

$$B = -Q^{-1} \cdot Q' \cdot Q^{-1}$$

$$\hat{Q}^{-1} = Q^{-1} - e \cdot (Q^{-1} \cdot Q' \cdot Q^{-1}) = Q^{-1}(1 - e \ Q' Q^{-1})$$

If

1. $Q' = Q'' \cdot Q$ and
2. Q is a unit quaternion ($N(Q) = 1, \ Q^{-1} = Q^*$), and
3. Q'' is a vector quaternion $(Q''^* = -Q'')$

then

$$\hat{Q}^{-1} = Q^* \cdot (1 - e \ Q'' \cdot Q \cdot Q^*) = Q^* \cdot (1 - e \ Q'')$$

$$= Q^* \cdot (1 + e \ Q''^*)$$

$$\hat{Q}^{-1} = \hat{Q}^*$$

This form echos the (plain) quaternion inverse in section 3.7.

```
dualquat_inv(q) := | qR ← dualquat_getReal(q)
                   | qRi ← quat_inv(qR)
                   | qD ← dualquat_getDual(q)
                   | Qtemp ← quat_mul(qRi, qD)
                   | Qtemp ← quat_mul(Qtemp, qRi)
                   | ret ← dualquat_quat(qRi, Qtemp)
                   | ret
```

4.7 Applications in kinematics - spatial transformations

The dual quaternion can be used to express the spatial transformations associated with screws. A screw motion is described as dual angular displacement about a dual vector axis. The dual angular displacement consists of a rotation (the 'real' part of the dual angle) about the screw axis and a translation (the 'dual' part of the dual angle) along the screw axis. The dual vector screw axis consists of a direction (the 'real' part of the dual vector) and its moment (the 'dual' part of the dual vector) about the origin.

Any rigid body displacement can be decomposed into a particular screw displacement about a particular screw axis.

4.7.1 Vector dual quaternion:

$$\hat{p} = \{0,\ \{p_x,\ p_y,\ p_z\}\} + e \cdot \{0, \{0,\ 0,\ 0\}\} = \{p + e \cdot 0\}$$

4.7.2 Dual spherical (rotation transform) quaternion:

$$\hat{Q} = \{\cos\frac{q}{2},\ \vec{\boldsymbol{n}} \cdot \sin\frac{q}{2}\} + e \cdot \{\ 0,\ \{0,\ 0,\ 0\}\} = \{R + e \cdot 0\}$$

4.7.3 Dual screw transform quaternion:

$$\hat{Q} = \left\{\cos\left(\frac{\hat{q}}{2}\right),\ \sin\left(\frac{\hat{q}}{2}\right)(\hat{\boldsymbol{s}})\right\}$$

The screw displacement is the dual angle $\hat{q} = q + e \cdot d$, along the screw axis defined by the dual vector $\hat{\boldsymbol{s}} = \vec{\boldsymbol{n}} + e \cdot \vec{\boldsymbol{m}}$

Recalling the trigonometric identities from section 1.7,

$$\hat{Q} = (\cos\frac{q}{2} - e \cdot \frac{d}{2} \cdot \sin\frac{q}{2},\ \vec{\boldsymbol{n}} \cdot \sin\frac{q}{2} + e \cdot (\vec{\boldsymbol{m}} \cdot \sin\frac{q}{2} + \vec{\boldsymbol{n}} \cdot \frac{d}{2} \cdot \cos\frac{q}{2})$$

$$\hat{Q} = \{\{\cos\frac{q}{2}, \{\sin\frac{q}{2} \cdot \vec{\boldsymbol{n}}\}\} + e \cdot \{-\frac{d}{2} \cdot \sin\frac{q}{2},\ \{\vec{\boldsymbol{m}} \cdot \sin\frac{q}{2} + \frac{d}{2} \cdot \vec{\boldsymbol{n}} \cdot \cos\frac{q}{2}\}\}\}$$

dualquat_screw(θ_hat, s_hat) :=

$\theta \leftarrow \theta\_hat_0$

$d \leftarrow \theta\_hat_1$

n ← dualquat_getReal(s_hat)

m ← dualquat_getDual(s_hat)

Q0 ← quat_rot(θ, n)

$Q1_0 \leftarrow -\left(\frac{d}{2}\right) \cdot \sin\left(\frac{\theta}{2}\right)$

for $i \in 1..3$

$\quad Q1_i \leftarrow m_i \cdot \sin\left(\frac{\theta}{2}\right) + \left(\frac{d}{2}\right) \cdot n_i \cdot \cos\left(\frac{\theta}{2}\right)$

dualquat_quat(Q0, Q1)

4.7.4 Combined translation along and rotation about an axis through the origin ($\vec{s} = \vec{n} + e \cdot 0$)

$$\hat{Q} = \{\{\cos\frac{q}{2}, \{\sin\frac{q}{2} \cdot \vec{n}\}\} + e \cdot \{-\frac{d}{2} \cdot \sin\frac{q}{2}, \{\frac{d}{2} \cdot \vec{n} \cdot \cos\frac{q}{2}\}\}$$

4.7.5 Pure rotation ($\hat{q} = q + e \cdot 0$) about an axis through the origin ($\vec{s} = \vec{n} + e \cdot 0$)

$$\hat{Q} = \{\{\cos\frac{q}{2}, \{\vec{n} \cdot \sin\frac{q}{2}\}\} + e \cdot \{0\}\}$$

4.7.6 Pure translation ($\hat{q} = 0 + e \cdot d$) along any axis ($\vec{s} = \vec{n} + e \cdot \vec{m}$)

$$\hat{Q} = \{\{1, \{ \vec{0} \}\} + e \cdot \{0, \{\frac{d}{2} \cdot \vec{n}\}\}\}$$

4.7.7 Dual spatial (rotation plus translation transform) quaternion:

$$\hat{Q} = \{R + e \cdot \frac{TR}{2}\}$$

Note: R is the unit rotation quaternion and T is the vector translation quaternion. In terms of the earlier general development, $R = Q$, $T = Q''$, so that $\hat{Q}^* = (1 + e\ T^*) \cdot Q^*$

```
dualquat_spatial(t,n,d) := | R ← quat_rot(t,n)
                           | T ← quat_mul_scalar(d,0.5)
                           | dualquat_quat(R,quat_mul(T,R))
```

```
dualquat_spatial_Bottema_p522(t,n,d) := | R ← quat_rot(t,n)
                                        | D0 ← 0.5 · (R1 · d1 + R2 · d2 + R3 · d3)
                                        | D1 ← 0.5 · (−R0 · d1 − R3 · d2 + R2 · d3)
                                        | D2 ← 0.5 · (R3 · d1 − R0 · d2 − R1 · d3)
                                        | D3 ← 0.5 · (−R2 · d1 + R1 · d2 − R0 · d3)
                                        | dualquat_quat(R,D)
```

4.7.5 General transform:

$\hat{P} = \hat{Q}_L \cdot \hat{p} \cdot \hat{Q}_R$, where $\hat{p} = \{p + e \cdot 0\}$ and $\hat{P} = \{P + e \cdot 0\}$ are dual vector quaternions representing points p and P in the moving and fixed coordinate systems, respectively, and Q_L , Q_R are dual transform quaternions described in the following sections dualquat_transform(Q, x) := dualquat_mul(dualquat_mul(Q, x), dualquat_conj(Q))

4.7.6 Pure rotation:

$\hat{Q}_L = \{R + e \cdot 0\}; \qquad \hat{Q}_R = \hat{Q}_L{}^* = \{R^* + e \cdot 0\}$

$\hat{P} = \{P + e \cdot 0\} = \{R + e \cdot 0\}\{p + e \cdot 0\}\{R^* + e \cdot 0\};$

$\hat{P} = \{R \cdot p \cdot R^* + e \cdot 0\}\ , \qquad P = R \cdot p \cdot R^*$

This quaternion transformation is equivalent to the common matrix transformation:

$$\begin{vmatrix} P \\ 1 \end{vmatrix} = \begin{vmatrix} R & 0 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} p \\ 1 \end{vmatrix}$$

where, here, P and p are 3-vectors, and R is the 3×3 rotation matrix.

4.7.7 Pure translation:

$\hat{Q}_L = \{1 + e \cdot \frac{T}{2}\} \qquad \hat{Q}_R = \hat{Q}_L{}^* = \{1 + e \cdot \frac{T^*}{2}\}$

$$\hat{P} = \{P + e \cdot 0\} = \{1 + e \cdot \frac{T}{2}\}\{p + e \cdot 0\}\{1 + e \cdot \frac{T^*}{2}\}$$

$$= \{p + e \cdot (\frac{T \cdot p}{2})\}\{1 + e \cdot \frac{T^*}{2}\} = \{p + e \cdot (\frac{T \cdot p}{2} + \frac{p \cdot T^*}{2})\}$$

$$p \cdot T^* = -p \cdot T = +T \cdot p$$

$$\hat{P} = (p + e \cdot \{T \cdot p\})$$

4.7.8 Combined rotation and translation:

$$\hat{Q}_L = \{R + e \cdot \frac{T \cdot R}{2}\} \qquad \hat{Q}_R = Q_L{*} = \{R{*} + e \cdot \frac{R{*} \cdot T{*}}{2}\}$$

$$\hat{P} = \{P + e \cdot 0\} = \{R + e \cdot \frac{T \cdot R}{2}\} \cdot \{p + e \cdot 0\} \cdot \{R{*} + e \cdot \frac{R{*} \cdot T{*}}{2}\}$$

$$= \{R \cdot p + e \cdot (\frac{T \cdot R \cdot p}{2})\} \cdot \{R{*} + e \cdot \frac{R{*} \cdot T{*}}{2}\}$$

$$\hat{P} = \{R \cdot p \cdot R{*} + e \cdot (\frac{T \cdot R \cdot p \cdot R{*}}{2} + \frac{R \cdot p \cdot R{*} \cdot T{*}}{2})\}$$

Because $(R \cdot p \cdot R{*})$ and T are vector quaternions,

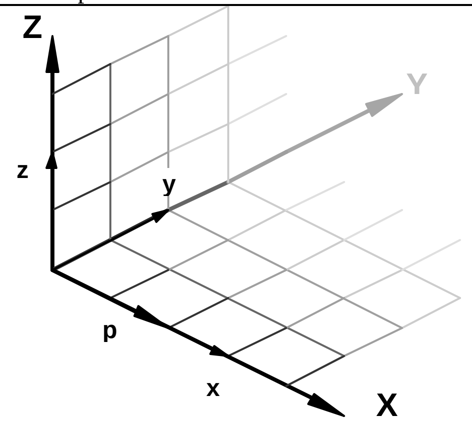
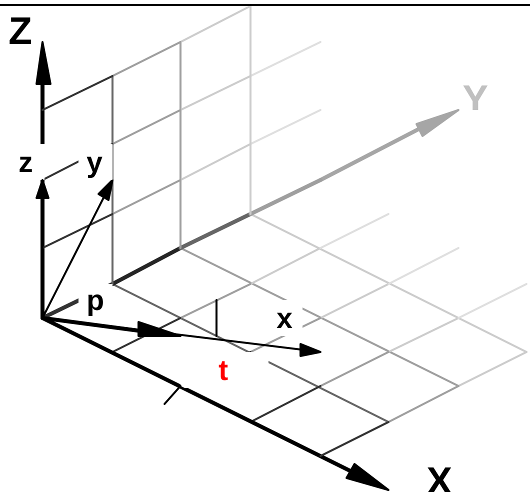
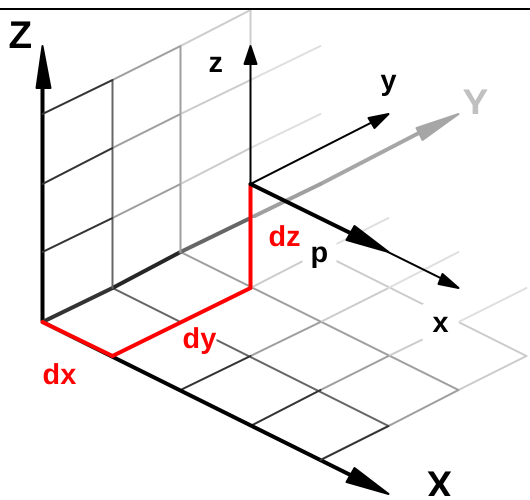
$$(R \cdot p \cdot R{*}) \cdot T{*} = -(R \cdot p \cdot R{*}) \cdot T = +T \cdot (R \cdot p \cdot R{*})$$

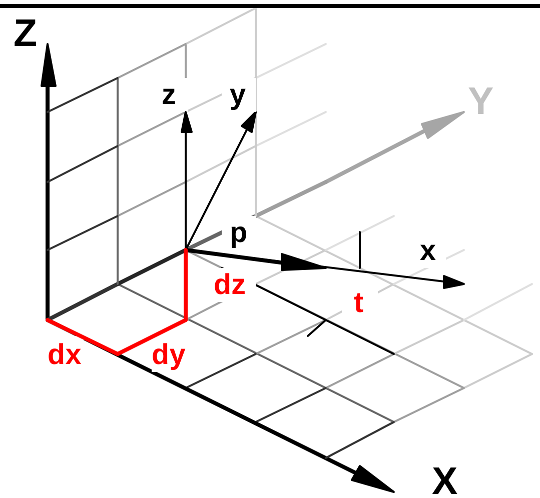
This dual quaternion transformation is equivalent to the common matrix transformation:

$$\begin{vmatrix} P \\ 1 \end{vmatrix} = \begin{vmatrix} R & T \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} p \\ 1 \end{vmatrix}$$

where, here, P and p are 3-vectors, R is the 3×3 rotation matrix, and T is the 3×1 translation matrix.

5. Discussion

Case	Description	Results
1		Case 1 - null transformation Rotation: $n := \text{quat\_vec}(0,0,1)$ $t := 0 \cdot \text{deg}$ Translation: $d := \text{quat\_vec}(0,0,0)$ Input vector: $p := \text{dualquat\_vec}(10,0,0)$ $p = \begin{pmatrix} 0 & 0 \\ 10 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ $P = \begin{pmatrix} 0 & 0 \\ 10 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ $R = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$
2		Case 2 - rotation 45 degrees about the z axis Rotation: $n := \text{quat\_vec}(0,0,1)$ $t := 45 \cdot \text{deg}$ Translation: $d := \text{quat\_vec}(0,0,0)$ Input vector: $p := \text{dualquat\_vec}(10,0,0)$ $p = \begin{pmatrix} 0 & 0 \\ 10 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ $P = \begin{pmatrix} 0 & 0 \\ 7.071 & 0 \\ 7.071 & 0 \\ 0 & 0 \end{pmatrix}$ $R = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$
3		Case 3 - translation (1,2,1.5) $n := \text{quat\_vec}(0,0,1)$ $t := 0 \cdot \text{deg}$ Translation: $d := \text{quat\_vec}(1,2,1.5)$ Input vector: $p := \text{dualquat\_vec}(10,0,0)$ $p = \begin{pmatrix} 0 & 0 \\ 10 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ $P = \begin{pmatrix} 0 & 0 \\ 10 & 0 \\ 0 & 15 \\ 0 & -20 \end{pmatrix}$ $R = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1.5 \end{pmatrix}$

4		Case 4 - translation (1,1,1) and 45 degrees rotation about z $n := \text{quat\_vec}(0,0,1)$ $t := 45 \cdot \text{deg}$ Translation: $d := \text{quat\_vec}(1,1,1)$ Input vector: $p := \text{dualquat\_vec}(10,0,0)$ $p = \begin{pmatrix} 0 & 0 \\ 10 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ $P = \begin{pmatrix} 0 & 0 \\ 7.071 & -7.071 \\ 7.071 & 7.071 \\ 0 & 1.776 \times 10^{-15} \end{pmatrix}$ $R = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$
5		
6		
7		
8		
9		
10		

6.0 References

6.1 O. Bottema & B. Roth, *Theoretical Kinematics*, Dover 1990, ISBN 0-486-66346-9

6.2 J. M. McCarthy, *Introduction to Theoretical Kinematics*, The MIT Press, ISBN 0262132524

6.3 Ian S. Fischer, *Dual-Number Methods in Kinematics, Statics, and Dynamics*, CRC Press, 1998, ISBN 0-8493-9115-6

7.0 Web Resources

7.1 David Lamb, *David Lamb's Clifford Algebra Primer*, `http://www.secs.oakland.edu/~latcha/spatial/clifford.html`

7.2 David Lamb, *Spatial Algebras in SOVAS* `http://www.secs.oakland.edu/~latcha/spatial/spatial.html`

7.3 David Lamb, *David Lamb's Screw Theory Primer* `http://www.secs.oakland.edu/~latcha/spatial/screw.html`

7.4 Michael Wagner, *Advanced Animation Techniques in VRML 97*, `http://vienna.eas.asu.edu/wagner/academic/vrml98/preface.html`